# Final Exam , MTH 221 , Summer 2018

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Solution by Yara Samer

$Score = \frac{50}{47}$

(i) (4 points) Find the solution set to the following system:

$$x_1 + x_2 + x_3 + x_4 = 0 \quad -x_1 - x_2 + x_4 = 2 \quad 2x_1 + 2x_2 + 3x_3 + 4x_4 = 4$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 2 \\ 2 & 2 & 3 & 4 & 4 \end{array}\right] \xrightarrow[-2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right] \xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

$x_1 + x_2 + x_3 + x_4 = 0$

$x_3 = 2 - x_4$

no solution as inconsistent

$0 = -2$

(ii) (4 points) Consider the following system $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -1 & -2 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c-a \end{bmatrix}$.

a. For what values of $a$, $b$, and $c$ will the system be inconsistent?

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 2 & 5 & 2 & b \\ -1 & -2 & c & c-a \end{array}\right] \xrightarrow[R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 0 & b-2a \\ 0 & 0 & 1+c & c \end{array}\right]$$

for $c = -1$

$a \in \mathbb{R}$

$b \in \mathbb{R}$

b. When it is consistent does the system have a unique solution or infinitely many solutions?

It will have a unique solution as $1 \neq 0$

(iii) (2 points) Determine the value(s) of $a$ so that the points $Q_1 = (1,0,5,0), Q_2 = (1,1,4,0), Q_3 = (1,4,a,0)$ are dependent.

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 1 & 1 & 4 & 0 \\ 1 & 4 & a & 0 \end{bmatrix} \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 4 & -5+a & 0 \end{bmatrix}$$

$\boxed{a = 1}$

(iv) (2 points) Convince me that

$$W = \{(a_1, a_2, a_3, a_4) \mid a_1 + a_2 \geq 0\}$$ $R^4$

is not a subspace of $R^4$.

$(5,6,1,1) \in W$ if $\alpha = -1$ $\quad -1(5,6,1,1) = (-5,-6,-1,-1)$

$5+6 = 11$

$a_1 + a_2 = -5 + -6 = -11 < 0$

fails second axiom under scalar multiplication.

(v) **(4 points)** Convince me that

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} a + b = 2c + d, \text{where } a, b, c, d \in R \right\}$$

is a subspace of $R^{2\times 2}$. Find the independent number of W (i.e., find dim(W) = IN(W)).

$R^{2\times 2} \sim R^4$

$a = 2c + d - b$

$W = \{(a, b, c, d) \mid a + b = 2c + d \mid a, b, c, d \in R\}$

$W = \{(2c + d - b, b, c, d) \mid a = 2c + d - b \mid b, c, d \in R\}$

$W = \{ b(-1, 1, 0, 0) + c(2, 0, 1, 0) + d(1, 0, 0, 1) \mid b, c, d \in R\}$

$W = \text{span}\{(-1, 1, 0, 0), (2, 0, 1, 0), (1, 0, 0, 1)\}$

$W = \text{Span}\left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ $\dim(W) = 3$

(vi) **(3 points)** Let $f_1, f_2, f_3, f_4, f_5$ be polynomials in $P_7$ such that each is of degree 3. Convince me that $f_1, f_2, f_3, f_4, f_5$ are dependent.

if each is of degree 3 then they live in $P_4$ and the dimension of $P_4$ is equal to 4 so the max independent polynomials that are of degree 3 can be 4 only. Here there is a fifth one which means dependent.

(vii) **(3 points)** Convince me that $P_3 = span\{f_1, f_2, f_3\}$ for some $f_1, f_2, f_3 \in P_3$ such that $degree(f_1) = degree(f_2) = degree(f_3) = 2$ (hint: Construct $f_1, f_2, f_3$) $P_3$ has dimension of 3 so span of 3 independent polynomials of degree 2 will equal to $P_3$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \underset{-R_1+R_2 \to R_3}{\overset{-R_1+R_2 \to R_2}{\sim}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ translate to $P_3$ so $P_3 = \text{span}\{x^2, x^2+x, x^2+1\}$

no free variables

3 leaders so 3 indep parts

dim = 3

(viii) **(6 points)** Let $T : R^4 \to R^3$ such that $T(a_1, a_2, a_3, a_4) = (a_1 + a_2, -a_1 - a_2, 0)$. Then $T$ is a linear transformation.

a. Find the standard matrix of $T$, call it $M$.

$$M = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

b. Find the zeros of $T$ (i.e., $Z(T)$) and find a basis for $Z(T)$

$$Z(T) = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$a_1 + a_2 = 0$

$a_1 = -a_2$

$Z(T) = \{(-a_2, a_2, a_3, a_4) \mid a_2, a_3, a_4 \in R\}$

$Z(T) = \{ a_2(-1, 1, 0, 0) + a_3(0, 0, 1, 0) + a_4(0, 0, 0, 1) \mid a_2, a_3, a_4 \in R\}$

Basis $= \{(-1, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$

c. Find the range of $T$ and find a basis for $Range(T)$

Range = Colmspace(T) = Span$\{(1, -1, 0)\}$

Basis $= \{(1, -1, 0)\}$

$A = QDQ^{-1}$

(xiii) **(6 points)** Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$. If $A$ is diagnosable, then find a diagonal matrix $D$ and an invertible matrix $Q$ such that $A = QDQ^{-1}$.

$(\alpha I_2 - A)Q^T = 0$

$C_{(A)} \quad |x I_2 - A| = 0$

$\left| \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \right| = \begin{vmatrix} \alpha - 2 & -1 \\ 0 & x - 4 \end{vmatrix} = 0$

$(\alpha - 2)(\alpha - 4) = 0$

$\alpha = 2 \quad \alpha = 4$

$E_2$

$Z(2I_2 - A) =$

$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & -2 & 0 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2}$

$\begin{array}{cc} x_1 & x_2 \end{array}$

$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} -x_2 = 0 \\ x_2 = 0 \end{array}$

$Z(2I_2 - A) \; \{(x_1, 0) \mid x_1 \in \mathbb{R}\}$

$E_2 = \text{span}\{(1, 0)\}$

$Z(4I_2 - A) =$

$E_4$ $\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} 2x_1 - x_2 = 0 \\ 2x_1 = x_2 \\ x_1 = \frac{1}{2}x_2 \end{array}$

$\text{Sol} = \{(\frac{1}{2}x_2, x_2) \mid x_2 \in \mathbb{R}\}$

$= \{x_2(\frac{1}{2}, 1) \mid x_2 \in \mathbb{R}\}$

$E_4 = \text{Span}\{(\frac{1}{2}, 1)\}$

$D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 0 \end{bmatrix}$

2

(xiv) **(3 points)** Let $A = \begin{bmatrix} 1 & a & b \\ -1 & 2 & c \\ -2 & -2a & 8 \end{bmatrix}$. For what values of $a, b, c$ will the matrix A be invertible?

$11 \neq 0$

$\begin{bmatrix} 1 & a & b \\ -1 & 2 & c \\ -2 & -2a & 8 \end{bmatrix} \xrightarrow[2R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2} \begin{bmatrix} 1 & a & b \\ 0 & a+2 & b+c \\ 0 & 0 & 2b+8 \end{bmatrix}$

$(1)(a+2)(2b+8) \neq 0$

$\boxed{a \neq -2}$ $\quad 2b + 8 \neq 0$

$2b = -8$

$\frac{2b}{2} = \frac{-8}{2}$

$\boxed{c \in \mathbb{R}}$ $\quad \boxed{b \neq -4}$

(xv) **(bonus, 3 points)** Let $A$ be a $3 \times 3$ matrix. Given 4 is an eigenvalue of $A$. Also given $(1, 0, 0) \in E_4$ and $(1, 0, 1) \in E_4$. Find the values of the first column of A and the values of the third column of A. (Note $E_4$ is the eigenspace of A that corresponds to the eigenvalue 4)

$\begin{bmatrix} 4 & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & \square \end{bmatrix}$

Solution on the back Page of Page 3

+3

(xvi) **(bonus, 1 point)** Who won the world cup last night?

France

+1

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(x) **(3 points)** Let $A$ be a $5 \times 5$ matrix such that $|A| = \pi$. Let $B = Col_1(A) + 3Col_5(A)$. Consider the system of equations $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = B$. Find the values of $x_1, x_3, x_5$

non zero

$$x_1 \begin{bmatrix} \\ \\ \\ \end{bmatrix} + x_2 \begin{bmatrix} \\ \\ \\ \end{bmatrix} + x_3 \begin{bmatrix} \\ \\ \\ \end{bmatrix} + x_4 \begin{bmatrix} \\ \\ \\ \end{bmatrix} + x_5 \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

1 2 3 4

1 3

uniqueness +

since $|A| \neq 0$ so the system is consistent indicating that $B$ can be written as a linear combination of columns of $A$.

$x_1 = 1$

$x_3 = 0$

$x_5 = 3$

(xi) **(4 points)** Let $M = span\{(1,0,1,1), (0,1,0,1), (0,1,1,0)\}$. Use Gram Schmidt algorithm to find an orthogonal basis for $M$.

$w_1 \quad w_2 \quad w_3$

$$\begin{matrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$

basis $= \{V_1, V_2, V_3\}$

$V_1 = (1,0,1,1)$

$$V_2 = W_2 - \frac{W_2 \cdot V_1}{|V_1|^2}(V_1) \rightarrow (0,1,0,1) - \frac{\overbrace{(0,1,0,1)\cdot(1,0,1,1)}^{1}}{\underbrace{(1,0,1,1)\cdot(1,0,1,1)}_{1+1+1=3}} \times (1,0,1,1)$$

$$V_3 = W_3 - \frac{W_3 \cdot V_2}{|V_2|^2}(V_2) - \frac{W_3 \cdot V_1}{|V_1|^2}(V_1)$$

$$V_3 = (0,1,1,0) - \frac{(0,1,1,0)\cdot(V_2)}{|V_2|^2} \times (V_2) - \frac{\overbrace{(0,1,1,0)\cdot(1,0,1,1)}^{1}}{\underbrace{(1,0,1,1)\cdot(1,0,1,1)}_{3}} \times (1,0,1,1)$$

$(\times V_2)$

$V_2 = \left((0,1,0,1) - \frac{1}{3}(1,0,1,1)\right) \times 3$

$V_2 = (0,3,0,3) - (1,0,1,1) = \underbrace{(-1,3,-1,2)}_{V_2}$

$V_3 = \left((0,1,1,0) - \frac{2}{15}(-1,3,-1,2) - \frac{1}{3}(1,0,1,1)\right) 15$

$= (0,15,15,0) - (-2,6,-2,4) - (5,0,5,5)$

$(-3, 9, 12, -9)$

(xii) **(3 points)** Let $A$ be a $3 \times 3$ matrix such that $2, -1, 3$ are the eigenvalues of $A$. Let $B = A^2 - A - 6I_3$. Find $|B|$.

$|A| = 2 \times -1 \times 3 = -6$

$|B| =$

Note that B = (A -3I_3)(A + 2I_3)
We know that the Eiginvalues of A - 3I_3 are 2 - 3 , -1 - 3, and 3 - 3 = -1, -4, 0. Eginenvalues of A + 2I_3 = 4, -1, 5
Let C = A - 3I_3, D = A + 2I_3. Then |C| = -1X4X0 = 0
|D| = 4X-1X5 = -20. Now |B| = |C||D| = 0X-20 = 0

$4 \to$ eigen value $\qquad E_4 = \text{span}\{(1,0,0),(1,0,1)\}$

Let $A$ $\underline{\underline{3 \times 3}}$

$$\underset{3\times3}{\begin{bmatrix} A \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = 4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{\text{1st column}} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$1\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} \qquad \underline{\text{3rd column}} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$A = \begin{array}{c} \begin{array}{ccc} \text{1st col} & & \text{3rd col.} \\ \downarrow & & \downarrow \end{array} \\ \begin{bmatrix} 4 & x & 0 \\ 0 & y & 0 \\ 0 & z & 4 \end{bmatrix} \end{array}$$